BRST charge for nonlinear algebras

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Abstract

We study the construction of the classical nilpotent canonical BRST charge for the nonlinear gauge algebras where a commutator (in terms of Poisson brackets) of the constraints is a finite order polynomial of the constraints.

1 Introduction

The BRST charge, corresponding to the Noether current of BRST symmetry [1], is one of the most efficient tools for studing the classical and quantum aspects of constrained systems. The properties of the BRST charge, especially its nilpotency, are the base of modern quantization methods of gauge theores in both Lagrangian [2] and Hamiltonian [3] formalism.

In this paper we discuss the form of the canonical BRST charge for a general enough class of gauge theories. Classical formulation of the gauge theory in phase space is characterized by first class constraints $T_\alpha = T_\alpha(p,q)$ with p_i and q^i being canonically conjugate phase variables. Constraints T_α satisfy the involution relations in terms of the Poisson bracket

$$\{T_\alpha, T_\beta\} = f^\gamma_{\alpha\beta} T_\gamma \tag{1}$$

with structure functions $f^\gamma_{\alpha\beta}$. In Yang-Mills type theories the structure functions are constants and the nilpotent BRST charge $\mathcal{Q}$ ($\{\mathcal{Q},\mathcal{Q}\} = 0$) can be written in a closed form. For general gauge theories the structure functions depend on phase variables $f^\gamma_{\alpha\beta} = f^\gamma_{\alpha\beta}(p,q)$ and the existence theorem for the nilpotent BRST charge has been proved [4]. It allows to present $\mathcal{Q}$ by series expansion (in general, infinite) in ghost variables

$$\mathcal{Q} = c^\alpha T_\alpha - \frac{1}{2} c^\alpha c^\beta f^\gamma_{\alpha\beta} \mathcal{P}_\gamma + \cdots = \mathcal{Q}_1 + \mathcal{Q}_2 + \cdots . \tag{2}$$

Here c^α and $\mathcal{P}_\alpha$ are canonically conjugate ghost variables and the dots mean the terms of higher orders in ghost variables conditioned by p, q dependence of structure functions. The problem, which we discuss here, consists in construction for a given constrained theory the higher order contributions to $\mathcal{Q}$ in terms of its structure functions. In general, solution to this problem is unknown.

We consider a class of gauge theories which can be described in terms of constraints T_α satisfying the relation (1) with nonconstant structure functions which form a finite order polynomial in the constraints T_α

$$f^\gamma_{\alpha\beta} = F^{\ \gamma}_{\alpha\beta} + V^{(1)\gamma\beta_1}_{\alpha\beta} T_{\beta_1} + \cdots + V^{(n-1)\gamma\beta_1\ldots\beta_{n-1}}_{\alpha\beta} T_{\beta_1} \cdots T_{\beta_{n-1}} \tag{3}$$

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where $F_{\alpha\beta}^{\ \ \gamma}, V_{\alpha\beta}^{(1)\gamma\beta_1}, ..., V_{\alpha\beta}^{(n-1)\gamma\beta_1...\beta_{n-1}}$ are constants. Algebras of such kind appeared to be in conformal field theories (the so-called $\mathcal{W}_N$ algebras) [5], in theories with quantum groups [6], in higher spin theories on AdS space [7].

Construction of the nilpotent BRST charge for quadratically nonlinear algebras ($V^{(2)} = ... = V^{(n-1)} = 0$) subjected to an additional special assumption concerning structure constants $V_{\alpha\beta}^{(1)\gamma\delta} = V_{\alpha\beta}^{\gamma\delta}$ was performed in [8] with the result

$$\mathcal{Q} = c^\alpha T_\alpha - \frac{1}{2}c^\alpha c^\beta F_{\alpha\beta}^\gamma \mathcal{P}_\gamma - \frac{1}{2}c^\alpha c^\beta V_{\alpha\beta}^\gamma \mathcal{P}_\gamma - \frac{1}{24}c^\alpha c^\beta c^\gamma c^\delta V_{\alpha\beta}^{\mu\nu} V_{\gamma\delta}^{\rho\sigma} F_{\mu\rho}^\lambda \mathcal{P}_\nu \mathcal{P}_\sigma \mathcal{P}_\lambda. \tag{4}$$

Notice that BRST analysis for quadratic algebras of different special kinds including the case considered in [8], was also given in [9]. As to general nonlinear algebras of the form (3), to our knowledge, the problem of construction of a nilpotent the BRST charge is open in this case.

In this paper we find some special restrictions on structure constants when the nilpotent BRST charge (2) can be presented in the simplest form including terms $\mathcal{Q}_1$ and $\mathcal{Q}_2$ only. The details are given in [10].

2 BRST charge for generic nonlinear algebras

Let us consider a theory with nonlinear algebras as described above (3). The structure constants $F_{\alpha\beta}^{\ \ \gamma}$ and $V_{\alpha\beta}^{(k-1)\alpha_1...\alpha_k}$ $(k = 2, 3, ..., n)$ are antisymmetric in lower indices and $V_{\alpha\beta}^{(k-1)\alpha_1...\alpha_k}$ $(k = 2, 3, ..., n)$ are totally symmetric in upper indices.

The Jacobi identities for these algebras have the form

$$F_{[\alpha\beta}^{\ \ \gamma} F_{\lambda]\gamma}^{\ \ \delta} = 0\ , \quad F_{[\alpha\beta}^{\ \ \rho} V_{\lambda]\rho}^{(1)\beta_1\beta_2} + V_{[\alpha\beta}^{(1)\rho(\beta_1} F_{\lambda]\rho}^{\ \ \beta_2)} = 0\ , \tag{5}$$

$$\begin{aligned} &F_{[\alpha\beta}^{\ \ \rho} V_{\lambda]\rho}^{(m)\beta_1...\beta_m\beta_{m+1}} + V_{[\alpha\beta}^{(m)\rho(\beta_1...\beta_m} F_{\lambda]\rho}^{\ \ \beta_{m+1})} + \\ &+ \sum_{k=1}^{m-1} C_{mk} V_{[\alpha\beta}^{(k)\rho(\beta_1...\beta_k} V_{\lambda]\rho}^{(m-k)\beta_{k+1}...\beta_{m+1})} = 0\ (m = 2, 3, ..., n-1)\ , \end{aligned} \tag{6}$$

$$\sum_{k=m-n+1}^{n-1} C_{mk} V_{[\alpha\beta}^{(k)\rho(\beta_1...\beta_k} V_{\lambda]\rho}^{(m-k)\beta_{k+1}...\beta_{m+1})} = 0\ (m = n, ..., 2n-2)\ , \tag{7}$$

where

$$C_{mk} = \frac{(k+1)!(m-k+1)!}{(m+1)!}. \tag{8}$$

In Eqs. (6), (7) symmetrization includes two sets of symmetric indices. We assume that in the symmetrization only one representative among equivalent ones obtained by permutation of indices into these sets is presented.

Contribution of the first order in ghost fields c^α, $\mathcal{Q}_1 = c^\alpha T_\alpha$, defines the nilpotency equation in the second order which has the solution

$$\mathcal{Q}_2 = -\frac{1}{2}c^\alpha c^\beta \Big(F_{\alpha\beta}^{\ \ \gamma} + \bar{V}_{\alpha\beta}^{\gamma\beta} T_\beta\Big)\mathcal{P}_\gamma. \tag{9}$$

Here the notation

$$\bar{V}_{\alpha\beta}^{\gamma\beta} = \sum_{k=1}^{n-1} V_{\mu\nu}^{(k)\alpha\beta\sigma_1...\sigma_{k-1}} T_{\sigma_1} \cdots T_{\sigma_{k-1}} \tag{10}$$

is used. Analyzing the nilpotency equation in the third order in ghost fields c^α we can find that if the following restrictions on structure constants of the algebra (3)

$$
\begin{aligned}
& V^{(k)\sigma\beta_1\sigma_1\ldots\sigma_{k-1}}_{[\alpha_1\alpha_2} V^{(m-k)\sigma_k\ldots\sigma_m}_{\alpha_3]\sigma} = 0, \\
& k = 1,2,...,n-1, \quad m > k, \quad m = 2,3,...,2n-2.
\end{aligned} \tag{11}
$$

are fulfilled, then the contribution to the BRST charge in the third order is equal to zero, $\mathcal{Q}_3 = 0$. Further analysis of the nilpotency equation in the forth order leads to the following contribution to the BRST charge

$$
\mathcal{Q}_4 = -\frac{1}{24} c^{\alpha_1} c^{\alpha_2} c^{\alpha_3} c^{\alpha_4} \Big(\bar{V}^{\sigma\beta_1}_{\alpha_1\alpha_2} \bar{V}^{\beta_2\rho}_{\alpha_3\alpha_4} F^{\beta_3}_{\sigma\rho} + 2\bar{V}^{\sigma\beta_1}_{\alpha_1\alpha_2} \tilde{V}^{\beta_2\rho}_{\alpha_3\alpha_4} F^{\beta_3}_{\sigma\rho} + \tilde{V}^{\sigma\beta_1}_{\alpha_1\alpha_2} \tilde{V}^{\beta_2\rho}_{\alpha_3\alpha_4} F^{\beta_3}_{\sigma\rho} \Big) \mathcal{P}_{\beta_1}\mathcal{P}_{\beta_2}\mathcal{P}_{\beta_3}, \tag{12}
$$

where the notation

$$
\tilde{V}^{\alpha\beta}_{\mu\nu} = \sum_{k=1}^{n-1} (k-1) V^{(k)\alpha\beta\sigma_1\ldots\sigma_{k-1}}_{\mu\nu} T_{\sigma_1}\cdots T_{\sigma_{k-1}}. \tag{13}
$$

was used. In the case of quadratically nonlinear algebras $\bar{V}^{\gamma\beta}_{\alpha\beta} = V^{(1)\gamma\beta}_{\alpha\beta}$, $\tilde{V}^{\alpha\beta}_{\mu\nu} = 0$ and from (12) it follows the result (4) in the forth order. The relations

$$
\bar{V}^{\sigma\beta_1}_{[\alpha_1\alpha_2} \bar{V}^{\rho(\beta_2}_{\alpha_3\alpha_4]} F^{\beta_3)}_{\sigma\rho} = 0, \quad \bar{V}^{\sigma[\beta_1}_{[\alpha_1\alpha_2} \tilde{V}^{\beta_2]\rho}_{\alpha_3\alpha_4]} F^{\beta_3}_{\sigma\rho} + \bar{V}^{\sigma[\beta_1}_{[\alpha_1\alpha_2} \tilde{V}^{\beta_3]\rho}_{\alpha_3\alpha_4]} F^{\beta_2}_{\sigma\rho} = 0, \tag{14}
$$

$$
\bar{V}^{\sigma\beta_1}_{[\alpha_1\alpha_2} \bar{V}^{\rho(\beta_2}_{\alpha_3\alpha_4]} \bar{V}^{\beta_3\lambda)}_{\sigma\rho} = 0, \quad \bar{V}^{\sigma[\beta_1}_{[\alpha_1\alpha_2} \tilde{V}^{\beta_2]\rho}_{\alpha_3\alpha_4]} \bar{V}^{\beta_3\lambda}_{\sigma\rho} + cycle(\beta_2,\beta_3,\lambda) = 0, \tag{15}
$$

$$
\tilde{V}^{\sigma\beta_1}_{[\alpha_1\alpha_2} \tilde{V}^{\rho(\beta_2}_{\alpha_3\alpha_4]} F^{\beta_3)}_{\sigma\rho} = 0, \quad \tilde{V}^{\sigma\beta_1}_{[\alpha_1\alpha_2} \tilde{V}^{\rho(\beta_2}_{\alpha_3\alpha_4]} \bar{V}^{\beta_3\lambda)}_{\sigma\rho} = 0 \tag{16}
$$

derived from the Jacobi identities (5) – (7) and the restrictions (11) were used to obtain the contribution (12). We point out that the restrictions (11) lead to equalities

$$
\bar{V}^{\sigma\beta_1}_{[\alpha_1\alpha_2} \bar{V}^{\beta_2\rho}_{\alpha_3\alpha_4]} \bar{V}^{\beta_3\lambda}_{\sigma\rho} = 0, \quad \tilde{V}^{\sigma\beta_1}_{[\alpha_1\alpha_2} \tilde{V}^{\beta_2\rho}_{\alpha_3\alpha_4]} \bar{V}^{\beta_3\lambda}_{\sigma\rho} = 0, \quad \bar{V}^{\sigma[\beta_1}_{[\alpha_1\alpha_2} \tilde{V}^{\beta_2]\rho}_{\alpha_3\alpha_4]} \bar{V}^{\beta_3\lambda}_{\sigma\rho} = 0. \tag{17}
$$

If now we additionally assume the following restrictions on the structure constants

$$
\begin{aligned}
& V^{(k)\beta\beta_1\sigma_1\ldots\sigma_{k-1}}_{[\alpha_1\alpha_2} V^{(m-k)\beta_2\gamma\sigma_k\ldots\sigma_{m-2}}_{\alpha_3\alpha_4]} F^{\beta_3}_{\beta\gamma} = 0, \\
& k = 1,...,n-1, \quad m > k, \quad m = 2,...,2n-2,
\end{aligned} \tag{18}
$$

then we have

$$
\bar{V}^{\sigma\beta_1}_{[\alpha_1\alpha_2} \bar{V}^{\beta_2\rho}_{\alpha_3\alpha_4]} F^{\beta_3}_{\sigma\rho} = 0, \quad \bar{V}^{\sigma[\beta_1}_{[\alpha_1\alpha_2} \tilde{V}^{\beta_2]\rho}_{\alpha_3\alpha_4]} F^{\beta_3}_{\sigma\rho} = 0, \quad \tilde{V}^{\sigma\beta_1}_{[\alpha_1\alpha_2} \tilde{V}^{\beta_2\rho}_{\alpha_3\alpha_4]} F^{\beta_3}_{\sigma\rho} = 0 \tag{19}
$$

and as the result $\mathcal{Q}_4 = 0$. Therefore there exists a unique form of the nilpotent BRST charge $\mathcal{Q} = \mathcal{Q}_1 + \mathcal{Q}_2$ if conditions (11), (18) are fulfilled. Although these conditions look like very restrictive, there exist the interesting algebras where they are fulfilled. For example, the conditions (11), (18) take place for Zamolodchikov's W_3 algebra with central extension and for the higher spin algebras in AdS space [7]. Of course, there exist non-linear algebras for which the conditions (11) and (18) are not fulfilled, e.g. these relations are not valid for $so(N)$-extended superconformal algebras with central extension [5] (see also [8]).

3 Summary

In this paper we have studied a construction of the nilpotent classical BRST charge for nonlinear algebras of the form (3) which are characterized by the structure constants $F_{\alpha\beta}^{\ \ \gamma}, V_{\alpha\beta}^{(1)\alpha_1\alpha_2}, ..., V_{\alpha\beta}^{(n-1)\alpha_1...\alpha_n}$. We have proved that if the conditions (11) and (18) are satisfied and a set of constraints T_α is linearly independent, the BRST charge is given in the universal form $\mathcal{Q} = \mathcal{Q}_1 + \mathcal{Q}_2$. Also we have proved that suitable quantities in terms of which one can efficiently analyze general nonlinear algebras (3) are $\bar{V}_{\alpha\beta}^{\mu\nu}, \tilde{V}_{\alpha\beta}^{\mu\nu}$.

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